ISRAEL JOURNAL OF MATHEMATICS **141** (2004), 221–233

GENERALIZED DEGREE AND OPTIMAL LOEWNER-TYPE INEQUALITIES

BY

SERGEI V. IVANOV*

Steklov Mathematical Institute
Fontanka 27, RU-191011 St. Petersburg, Russia
e-mail: svivanov@pdmi.ras.ru

AND

MIKHAIL G. KATZ**

Department of Mathematics and Statistics, Bar Ilan University
Ramat Gan 52900, Israel
e-mail: katzmik@math.biu.ac.il

ABSTRACT

We generalize optimal inequalities of C. Loewner and M. Gromov, by proving lower bounds for the total volume in terms of the homotopy systole and the stable systole. Our main tool is the construction of an area-decreasing map to the Jacobi torus, streamlining and generalizing the construction of the first author in collaboration with D. Burago. It turns out that one can successfully combine this construction with the coarea formula, to yield new optimal inequalities.

1. Loewner's and Gromov's optimal inequalities

Over half a century ago, C. Loewner proved that the least length $\mathrm{sys}_1(\mathbb{T}^2)$ of a noncontractible loop on a Riemannian 2-torus $\mathbb{T}^2$ satisfies the optimal inequality

$$\mathrm{sys}_1(\mathbb{T}^2)^2 \le \gamma_2 \,\mathrm{area}(\mathbb{T}^2), \tag{1.1}$$

* Supported by grants CRDF RM1-2381-ST-02, RFBR 02-01-00090 and NS-1914.2003.1.

** Supported by the Israel Science Foundation (grants no. 620/00-10.0 and 84/03).

Received November 4, 2003

where $\gamma_2 = 2/\sqrt{3}$, cf. [Pu52, CK03].

An optimal generalisation of Loewner's inequality is due to M. Gromov [Gr99, pp. 259–260] (cf. [CK03, inequality (5.14)]) based on the techniques of D. Burago and S. Ivanov [BI94, BI95]. We give below a slight generalisation of Gromov's statement.

Definition 1.1: Given a map $f\colon X \to Y$ between closed manifolds of the same dimension, we denote by $\deg(f)$ either the algebraic degree of f when both manifolds are orientable, or the absolute degree, otherwise.

We denote by $\mathcal{A}_X$ the Abel-Jacobi map of X, cf. formula (2.1); by γ_n, the Hermite constant, cf. formula (5.1); and by $\operatorname{stsys}_1(g)$, the stable 1-systole of a metric g, cf. formula (5.3).

THEOREM 1.2 (M. Gromov): *Let X^n be a compact manifold of equal dimension and first Betti number: $\dim(X) = b_1(X) = n$. Then every metric g on X satisfies the following optimal inequality:*

$$\deg(\mathcal{A}_X)\operatorname{stsys}_1(g)^n \le (\gamma_n)^{n/2}\operatorname{vol}_n(g). \tag{1.2}$$

The boundary case of equality in inequality (1.2) is attained by flat tori whose group of deck transformations is a critical lattice in $\mathbb{R}^n$.

Note that the inequality is nonvacuous in the orientable case only if the cuplength of X is n, i.e. the Abel–Jacobi map $\mathcal{A}_X$ is of nonzero algebraic degree. Recall that a critical lattice (i.e. one attaining the value of the Hermite constant (5.1)) is in particular extremal, and being extremal for a lattice implies perfection and eutaxy [Bar57].

Having presented the results of Loewner and Gromov, we now turn to their generalisations, described in Sections 2 and 3.

Our main tool is the construction of an area-decreasing map to the Jacobi torus, streamlining and generalizing the construction of [BI94], cf. [Gr99]. An important theme of the present work is the observation that one can successfully combine this construction with the coarea formula, to yield new optimal inequalities. Historical remarks and a discussion of the related systolic literature can be found at the end of the next section.

2. Generalized degree and first theorem

Let (X, g) be a Riemannian manifold. Let $n = \dim(X)$ and $b = b_1(X)$. Loewner's inequality can be generalized by optimal inequalities in two different ways, as illustrated in Figure 2.1, where the constants γ_n and γ'_n are defined in Section 5, while

$$\mathcal{A}_X\colon X \to \mathbb{T}^b \tag{2.1}$$

is the Abel–Jacobi map ([Li69, Gr96], cf. [BK03$_B$, (4.3)]) inducing isomorphism in 1-dimensional cohomology.

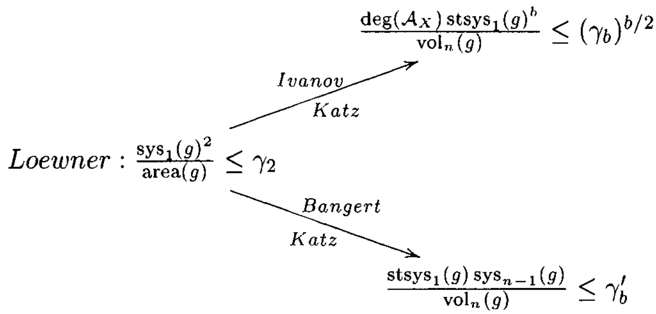

Figure 2.1. Two generalisations of Loewner's theorem, cf. (2.3) and (2.5).

The map $\mathcal{A}_X\colon X \to T^b$ lifts to a proper map

$$\overline{\mathcal{A}_X}\colon \overline{X} \to \mathbb{R}^b, \tag{2.2}$$

where $\mathbb{R}^b$ is the universal cover of the Jacobi torus. The corresponding fibers of $\mathcal{A}_X$ and $\overline{\mathcal{A}_X}$ (i.e. fibers which project to the same point of T^b) are homeomorphic. On the other hand, the nonvanishing of the homology class of the typical fiber of $\overline{\mathcal{A}_X}$ in $H_{n-b}(\overline{X})$ is far weaker than the nonvanishing of the homology class of the typical fiber of $\mathcal{A}_X$ in $H_{n-b}(X)$ (and thus the resulting theorem is more interesting). For instance, for the standard nilmanifold of the Heisenberg group (cf. Remark 2.5) we have $[\overline{X}] \neq 0$, but the homology class of the fiber of $\mathcal{A}_X$ is trivial. The latter is true whenever $n = b + 1$.

In the theorems below, to obtain a nonvacuous inequality, we replace the condition of nonvanishing degree in Gromov's theorem by the nonvanishing of the homology class $[\overline{X}]$ of the lift of the typical fiber of $\mathcal{A}_X$ to the universal abelian cover $\overline{X}$ of X, cf. [Gr83, p. 101] and equation (2.2) below. In particular, we must have $b \le n$ to obtain a nonvacuous inequality. Following M. Gromov [Gr83, p. 101], we introduce the following notion of generalized degree.

Definition 2.1: Denote by $\deg(\mathcal{A}_X)$ the infimum of $(n-b)$-volumes of integral cycles representing the class $[\overline{X}] \in H_{n-b}(\overline{X})$, if X is orientable, and infimum of volumes of $\mathbb{Z}_2$-cycles, otherwise.

Remark 2.2: This quantity is denoted 'deg' in [Gr83, p. 101]. When the dimension and the first Betti number coincide, this quantity is a topological invariant. In general, of course, it is not. Yet it is remarkable that the nonvanishing of this quantity suffices to produce a nonvacuous volume lower bound of Theorem 1.3, generalizing Gromov's theorem 1.2.

THEOREM 2.3: *Let X be a compact manifold. Let $n = \dim(X)$ and $b = b_1(X)$, and assume $b \geq 1$. Then every metric g on X satisfies the inequality*

$$\deg(\mathcal{A}_X)\, \mathrm{stsys}_1(g)^b \leq (\gamma_b)^{b/2} \mathrm{vol}_n(g). \tag{2.3}$$

Here the stable 1-systole $\mathrm{stsys}_1(g)$ is defined in formula (5.3). In particular, for $n = b + 1$ we obtain the following corollary.

COROLLARY 2.4: *Let X be a compact manifold. Let $b = b_1(X)$. Assume that $\dim(X) = b + 1$, and $[\overline{X}] \neq 0$. Then every metric g on X satisfies the following optimal inequality:*

$$\mathrm{stsys}_1(g)^b\, \mathrm{sys}\,\pi_1(g) \leq (\gamma_b)^{b/2}\, \mathrm{vol}_{b+1}(g), \tag{2.4}$$

where $\mathrm{sys}\,\pi_1(g)$ denotes the least length of a shortest noncontractible loop for the metric g.

Remark 2.5: To give an example where equality is attained in inequality (2.4), it suffices to take a Riemannian fibration by circles of constant length, over a flat torus whose group of deck tranformations is critical (choose the circles sufficiently short, so as to realize the value of the invariant $\mathrm{sys}\,\pi_1(g)$). This can be realized, for instance, by compact quotients of left invariant metrics on the 3-dimensional Heisenberg group (here $b = 2$), cf. [Gr83, p. 101].

The results of the present paper are further generalized in [BCIK]. A nonsharp version of inequalities (2.4) and (4.1) for an arbitrary pair $b \leq n$ was proved using different techniques in [KKS], resulting in an inequality with an extra multiplicative constant $C(n)$ depending on the dimension but not on the metric. Note that a different sharp generalisation of Loewner's inequality was studied in [BK03$_A$]:

$$\mathrm{stsys}_1(g)\, \mathrm{sys}_{n-1}(g) \leq \gamma'_b\, \mathrm{vol}_n(g), \tag{2.5}$$

where γ'_b is the Bergé–Martinet constant, cf. (5.2). The work [BK03$_B$] studies the boundary case of equality of a further generalisation of inequality (2.5).

We would like to point out an interesting difference between inequalities (2.4) and (2.5). Namely, inequalities for the 1-systole, such as (2.4), tend to be satified even by the ordinary (i.e. unstable) systole, cf. [Gr96, $3C_1$] and [CK03, (3.5)], albeit with a nonsharp constant. Meanwhile, inequality (2.5) for ordinary systoles is definitely violated by a suitable sequence of metrics, no matter what the constant, cf. [BaK98]. Volume lower bounds in terms of systoles and the study of the associated constants for 4-manifolds appear in [Ka03].

The work [CK03] surveys other universal (curvature-free) volume bounds and formulates a number of open questions, including the one about the existence of such a lower bound in terms of the least length of a nontrivial closed geodesic (perhaps contractible) on X, for any manifold X, cf. [NR02, Sa04].

3. Pu's inequality and generalisations

To state the next theorem, we need to recall the inequality of P. Pu. We record here a slight generalisation of the inequality from [Pu52]; see also [Iv02] for an alternative proof and generalisations. Namely, every surface (S, g) which is not a 2-sphere satisfies

$$\operatorname{sys}\pi_1(g)^2 \leq \frac{\pi}{2}\operatorname{area}(g), \tag{3.1}$$

where the boundary case of equality in (3.1) is attained precisely when, on the one hand, the surface S is a real projective plane, and on the other, the metric g is of constant Gaussian curvature.

The generalisation follows from Gromov's inequality (3.2) below (by comparing the numerical values of the two constants). Namely, every aspherical compact surface (S, g) admits a metric ball

$$B = B_p(\tfrac{1}{2}\operatorname{sys}\pi_1(g)) \subset S$$

of radius $\frac{1}{2}\operatorname{sys}\pi_1(g)$ which satisfies [Gr83, Corollary 5.2.B]

$$\operatorname{sys}\pi_1(g)^2 \leq \frac{4}{3}\operatorname{area}(B). \tag{3.2}$$

Let S be a nonorientable surface, and let $\phi\colon \pi_1(S) \to \mathbb{Z}_2$ be an epimorphism from its fundamental group to $\mathbb{Z}_2$, corresponding to a map $\hat{\phi}\colon S \to \mathbb{R}P^2$ of absolute degree +1. We define the "1-systole relative to ϕ", denoted $\phi\operatorname{sys}_1(g)$, of a metric g on S, by minimizing length over loops γ which are not in the kernel of ϕ, i.e. loops whose image under ϕ is not contractible in the projective plane:

$$\phi\operatorname{sys}_1(g) = \min_{\phi([\gamma])\neq 0 \in \mathbb{Z}_2} \operatorname{length}(\gamma). \tag{3.3}$$

Does every nonorientable surface (S, g) and map $\hat{\phi}\colon S \to \mathbb{R}P^2$ of absolute degree one satisfy the following relative version of Pu's inequality:

$$\phi \operatorname{sys}_1(g)^2 \leq \frac{\pi}{2} \operatorname{area}(g)? \tag{3.4}$$

This inequality is related to Gromov's (non-sharp) inequality $(*)_{inter}$ from [Gr96, 3.C.1]; see also [Gr99, Theorem 4.41]. This question appeared in [CK03, conjecture 2.7]. Let

$$\sigma_2 = \sup_{(S,g)} \frac{\phi \operatorname{sys}_1(g)^2}{\operatorname{area}(g)}, \tag{3.5}$$

where the supremum is over all nonorientable surfaces S, metrics g on S, as well as maps ϕ as above. Thus we ask whether $\sigma_2 = \pi/2$. Calculating σ_2 depends on calculating the filling area of the Riemannian circle, cf. Proposition 3.1 below.

PROPOSITION 3.1: *We have the following estimate: $\sigma_2 \in [\pi/2, 2]$.*

Proof: If we open up a surface S as above along a shortest essential loop γ (in the sense that $\phi([\gamma]) \neq 0$), we obtain a $\mathbb{Z}_2$-filling Σ (possibly nonorientable) of a circle of length $2\operatorname{length}(\gamma)$. It is clear that the boundary circle is imbedded in Σ isometrically as a metric space. Thus it suffices to prove that that the filling area of a circle of length 2π is at least $\pi^2/2$. Choose two points on the boundary circle at distance $\pi/2$ from each other. Consider the map $\Sigma \to \mathbb{R}^2$ whose coordinate functions are the distances to these points. The map is area-decreasing, and its image contains a square of area $\pi^2/2$ (encircled by the image of the boundary), namely the Pythagorean "diamond" inside the square $[0, \pi] \times [0, \pi]$ in the plane. ∎

4. Second theorem: the case $\dim(X) = b_1(X) + 2$

THEOREM 4.1: *Let X be a compact nonorientable manifold. Let $b = b_1(X)$. Assume $\dim(X) = b + 2$ and $\iota[\overline{X}] \neq 0$, where $\iota\colon H_2(\overline{X}, \mathbb{Z}_2) \to H_2(K, \mathbb{Z}_2)$ is the homomorphism induced by a map to some aspherical space K. Then every metric g on X satisfies the following inequality:*

$$\operatorname{stsys}_1(g)^b \operatorname{sys}\pi_1(g)^2 \leq \sigma_2 \gamma_b^{b/2} \operatorname{vol}_{b+2}(g), \tag{4.1}$$

where σ_2 is the optimal systolic ratio from (3.5).

The proof appears at the end of Section 8.

Remark 4.2: If, as conjectured, we have $\sigma_2 = \pi/2$, then the boundary case of equality in inequality (4.1) is attained by Riemannian submersions over a flat critical torus, with minimal fibers isometric to a fixed real projective plane with a metric of constant Gaussian curvature.

Example 4.3: For $X = \mathbb{R}P^2 \times \mathbb{T}^2$, we obtain the following inequality:

$$\operatorname{stsys}_1(g)^2 \operatorname{sys}\pi_1(g)^2 \leq \sigma_2\gamma_2 \operatorname{vol}_4(g),$$

which can be thought of as a "Pu-times-Loewner" inequality, cf. (1.1) and (3.1) (particularly if we prove that $\sigma_2 = \pi/2$).

5. Lattices, Hermite and Bergé–Martinet constants

Given a lattice $L \subset (B, \|\cdot\|)$ in a Banach space B with norm $\|\cdot\|$, denote by $\lambda_1(L) = \lambda_1(L, \|\cdot\|) > 0$ the least norm of a nonzero vector in L. Then the Hermite constant $\gamma_n > 0$ is defined by the supremum

$$\sup_{L \subset \mathbb{R}^n} \frac{\lambda_1(L)^n}{\operatorname{vol}(\mathbb{R}^n/L)} = (\gamma_n)^{n/2}, \tag{5.1}$$

where the supremum is over all lattices with respect to a Euclidean norm. The particular choice of the exponent $n/2$ may be motivated by the *linear* asymptotic behavior of γ_n as a function of $n \to \infty$, cf. [LLS90, pp. 334 and 337].

A related constant γ'_b, called the Bergé–Martinet constant, is defined as follows:

$$\gamma'_b = \sup\{\lambda_1(L)\lambda_1(L^*) | \ L \subseteq \mathbb{R}^b\}, \tag{5.2}$$

where the supremum is over all Euclidean lattices L. Here L^* is the lattice dual to L. If L is the $\mathbb{Z}$-span of vectors (x_i), then L^* is the $\mathbb{Z}$-span of a dual basis (y_j) satisfying $\langle x_i, y_j \rangle = \delta_{ij}$.

In a Riemannian manifold (X, g), we define the volume $\operatorname{vol}_k(\sigma)$ of a Lipschitz k-simplex $\sigma\colon \Delta^k \to X$ to be the integral over the k-simplex Δ^k of the "volume form" of the pullback $\sigma^*(g)$. The stable norm $\|h\|$ of an element $h \in H_k(X, \mathbb{R})$ is the infimum of the volumes $\operatorname{vol}_k(c) = \Sigma_i |r_i| \operatorname{vol}_k(\sigma_i)$ over all real Lipschitz cycles $c = \Sigma_i r_i \sigma_i$ representing h. We define the stable 1-systole of the metric g by setting

$$\operatorname{stsys}_1(g) = \lambda_1(H_1(X, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|), \tag{5.3}$$

where $\|\cdot\|$ is the stable norm in homology associated with the metric g.

6. A decomposition of the John ellipsoid

The following statement may be known by convex set theorists. A proof may be found in in [BI94]. Recall that the John ellipsoid of a convex set in Euclidean space is the unique ellipsoid of largest volume inscribed in it [MS86].

LEMMA 6.1: *Let $(V^d, \|\cdot\|)$ be a Banach space. Let $\|\cdot\|_E$ be the Euclidean norm determined by the John ellipsoid of the unit ball of $\|\cdot\|$. Then there exists a decomposition of $\|\cdot\|_E^2$ into rank-1 quadratic forms:*

$$\|\cdot\|_E^2 = \sum_{i=1}^{N} \lambda_i L_i^2$$

such that $N \leq d(d+1)/2+1$, $\lambda_i > 0$ for all i, $\sum \lambda_i = d$, and L_i: $V \to \mathbb{R}$ are linear functions with $\|L_i\|^* = 1$ where $\|\cdot\|^*$ is the dual norm to $\|\cdot\|$.

7. An area-nonexpanding map

Let X be a compact Riemannian manifold, Y a topological space, and let φ: $X \to Y$ be a continuous map inducing an epimorphism in one-dimensional real homology. Then one defines the *relative stable norm* $\|\cdot\|_{st/\varphi}$ on $H_1(Y;\mathbb{R})$ by

$$\|\alpha\|_{st/\varphi} = \inf\{\|\beta\|_{st} : \beta \in H_1(X;\mathbb{R}),\ \varphi_*(\beta) = \alpha\},$$

where $\|\cdot\|_{st}$ is the ordinary ("absolute") stable norm. The stable norm itself may be thought of as the relative stable norm defined by the Abel–Jacobi map to the torus $H_1(X,\mathbb{R})/H_1(X,\mathbb{Z})_{\mathbb{R}}$.

Definition 7.1: We will say that a Lipschitz map $\mathcal{A}$: $X \to M$ between Riemannian manifolds is "non-expanding on all d-dimensional areas" if for every smooth d-dimensional submanifold Y of X, one has $\mathrm{vol}_d(\mathcal{A}(Y)) \leq \mathrm{vol}_d(Y)$.

Equivalently, $\mathrm{Jac}(\mathcal{A}|_Y) \leq 1$ wherever $\mathcal{A}|_Y$ is differentiable.

Let X^n be a compact Riemannian manifold, V^d a vector space and Γ a lattice in V. We will identify V and $H_1(V/\Gamma;\mathbb{R})$.

PROPOSITION 7.2: *Let φ: $X \to V/\Gamma$ be a continuous map inducing an epimorphism of the fundamental groups and $\|\cdot\|_E$ denote the Euclidean norm on V defined by the John ellipsoid of the relative stable norm $\|\cdot\|_{st/\varphi}$. Then there exists a Lipschitz map $\mathcal{A}$: $X \to (V/\Gamma, \|\cdot\|_E)$ which is homotopic to φ and non-expanding on all d-dimensional areas, where $d = \dim(V)$.*

The proof of Propostion 7.2 appears at the end of this section. There is a natural isomorphism $\Gamma \simeq \pi_1(V/\Gamma) \simeq \pi_1(X)/\ker(\varphi_*)$. Consider a covering space

$\overline{X}$ of X defined by the subgroup $\ker(\varphi_*)$ of $\pi_1(X)$. Then Γ acts on $\overline{X}$ as the deck group $\pi_1(X)/\ker(\varphi_*)$. This action will be written additively, as in (7.1) below. It is sufficient to construct a Lipschitz map $\overline{X} \to V$ which is Γ-equivariant and does not expand d-dimensional areas. We need the following lemma.

LEMMA 7.3: *For every linear function $L\colon V \to \mathbb{R}$ such that $\|L\|^*_{st/\varphi} = 1$ there exists a 1-Lipschitz function $f\colon \overline{X} \to \mathbb{R}$ such that*

$$f(x+v) = f(x) + L(v) \tag{7.1}$$

for all $x \in \overline{X}$ and $v \in \Gamma$.

Proof: Fix an $x_0 \in \overline{X}$, consider the orbit $\overline{X}_0 = \{x_0 + v : v \in \Gamma\}$ and define a function $f_0\colon \overline{X}_0 \to \mathbb{R}$ by $f_0(x_0 + v) = L(v)$. Note that f_0 satisfies (7.1) for $x \in \overline{X}_0$. For every $v \in \Gamma$ and $x \in \overline{X}$, one has $L(v) \leq \|v\|_{st/\varphi}$ and $\|v\|_{st/\varphi}$ is no greater than the distance between x and $x + v$. Hence f_0 is 1-Lipschitz . Every 1-Lipschitz function defined on a subset of a metric space admits a 1-Lipschitz extension to the whole space, by the triangle inequality. Moreover, an extension can be chosen so that the equivariance (7.1) is preserved. For example, we can set $f(x) = \inf\{f_0(y) + |xy| : y \in \overline{X}_0\}$, where $|xy|$ denotes the distance. ∎

Proof of Proposition 7.2: Applying Lemma 6.1 to the norm $\|\cdot\|_{st/\varphi}$ yields a decomposition

$$\|\cdot\|_E^2 = \sum_{i=1}^N \lambda_i L_i^2,$$

where $\lambda_i > 0$, $\sum \lambda_i = d$, $L_i \in V^*$ and $\|L_i\|^*_{st/\varphi} = 1$. Then a linear map $L\colon V \to \mathbb{R}^N$ defined by

$$L(x) = (\lambda_1^{1/2} L_1(x), \lambda_2^{1/2} L_2(x), \dots, \lambda_N^{1/2} L_N(x))$$

is an isometry from $(V, \|\cdot\|_E)$ onto a subspace $L(V)$ of $\mathbb{R}^N$, equipped with the restriction of the standard coordinate metric of $\mathbb{R}^N$.

By Lemma 7.3, for every $i = 1, 2, \dots, N$ there exists a 1-Lipschitz function $f_i\colon \overline{X} \to \mathbb{R}$ such that $f_i(x + v) = f_i(x) + L_i(v)$ for all $x \in \overline{X}$ and $v \in \Gamma$. Define a map $F\colon \overline{X} \to \mathbb{R}^N$ by

$$F(x) = (\lambda_1^{1/2} f_1(x), \lambda_2^{1/2} f_2(x), \dots, \lambda_N^{1/2} f_N(x)). \tag{7.2}$$

Observe that both L and F are Γ-equivariant with respect to the following action of Γ on $\mathbb{R}^N$:

$$\Gamma \times \mathbb{R}^N \to \mathbb{R}^N, \quad (v, x) \mapsto x + L(v).$$

Now let $\Pr_{L(V)}: \mathbb{R}^N \to L(V)$ be the orthogonal projection to $L(V)$. Then the composition $L^{-1} \circ \Pr_{L(V)} \circ F$ is a Γ-equivariant map from $\overline{X}$ to V. Since the projection is nonexpanding and L is an isometry, it suffices to prove that the map F of (7.2) is nonexpanding on d-dimensional areas.

Let Y be a smooth d-dimensional submanifold of $\overline{X}$. Since F is Lipschitz, the restriction $F|_Y$ is differentiable a.e. on Y. Let $y \in Y$ be such that $F|_Y$ is differentiable at y, and let $A = d(F|_Y): T_yY \to \mathbb{R}^N$. Then we obtain

$$\operatorname{trace}(A^*A) = \sum \lambda_i |d(f_i\,|_Y)|^2 \leq \sum \lambda_i = d$$

since the functions f_i are 1-Lipschitz . By the inequality of geometric and arithmetic means, we have

$$\begin{aligned}\operatorname{Jac}(F|_Y)(x) &= \det(A^*A)^{1/2}\\ &\leq \left(\frac{1}{d}\operatorname{trace}(A^*A)\right)^{d/2}\\ &\leq 1,\end{aligned}$$

proving the proposition. ∎

8. Proof of Theorem 2.3 and Theorem 4.1

Consider the Jacobi torus $J_1(X) = H_1(X,\mathbb{R})/H_1(X,\mathbb{Z})_{\mathbb{R}}$ and the Abel–Jacobi map $\mathcal{A}_X: X \to J_1(X)$ constructed in Proposition 7.2.

Remark 8.1: The map $\mathcal{A}_X$ can be replaced by a smooth one by an arbitrarily small perturbation, in such a way as to expand d-dimensional areas at most by a factor $1+\epsilon$. Our main tool will be the coarea formula; see below. We can also avoid the above smoothing argument, and use instead the current-theoretic version of the coarea formula, relying on H. Federer's theory of "slicing". Given an integral current T in a smooth oriented manifold M (e.g. in our case $T = [M]$), and a Lipschitz map $f\colon M \to N$, one can in a sense decompose T by f, obtaining currents (slices) $\langle T, f, y\rangle$ supported in the fibers $f^{-1}(y)$, for a.e. $y \in N$. If T is a cycle then so are the slices; if T is a submanifold and f is smooth, then the slices at regular values of f are just (integration over) the typical fibers of the map. The main properties of this operation are given in [Fe69, Thm 4.3.2, p. 438]. In particular, item 4.3.2(2) is a version of the coarea formula. In section 4.4, Federer shows that the usual homology groups of a reasonably good space (e.g. manifold in our case) coincide with the ones defined via currents.

Proof of Theorem 2.3: We exploit the coarea formula [Fe69, 3.2.11], [Ch93, p. 267] as in [Gr83, Theorem 7.5.B].

Away from the negligible singular set, the smooth map is a submersion. Therefore the metric on X can be modified by a volume-preserving deformation so that the map actually becomes *distance* decreasing, up to an arbitrarily small amount. Note, however, that the coarea formula could be applied even without replacement by a short map. Formally we only need two facts:

1. if the map is area-nonincreasing, then the volume is no smaller than the area of the image times the minimal area of a fiber,
2. almost every fiber is "typical" and hence has area no less than the generalized degree $\deg(\mathcal{A}_X)$, cf. inequality (8.1).

Let $S = \mathcal{A}_X^{-1}(p)$ be the surface which is a regular fiber of least $(n-b)$-volume. (If there is none, choose S to be within $\epsilon > 0$ of the infimum, and then let $\epsilon \to 0$.) Then

$$\deg(\mathcal{A}_X)\,\mathrm{stsys}_1(g)^b \leq \mathrm{vol}_{n-b}(S)\,\mathrm{stsys}_1(g)^b. \tag{8.1}$$

Note that $\mathcal{A}_X$ induces isometry in $H_1(,\mathbb{R})$ with respect to the stable norm of the metric g. Hence

$$\deg(\mathcal{A}_X)\,\mathrm{stsys}_1(g)^b \leq \mathrm{vol}_{n-b}(S)\,\mathrm{stsys}_1(J_1(X), \|\cdot\|)^b.$$

We now replace the stable norm $\|\cdot\|$ by the flat Euclidean metric $\|\cdot\|_E$ defined by the John ellipsoid of the stable norm:

$$\deg(\mathcal{A}_X)\,\mathrm{stsys}_1(g)^b \leq \mathrm{vol}_{n-b}(S)\,\mathrm{stsys}_1(J_1(X), \|\cdot\|_E)^b.$$

By definition of the Hermite constant,

$$\deg(\mathcal{A}_X)\,\mathrm{stsys}_1(g)^b \leq \mathrm{vol}_{n-b}(S)\gamma_b^{b/2}\,\mathrm{vol}_b(J_1(X), \|\cdot\|_E).$$

Now we apply the coarea formula to our map which is decreasing on b-dimensional volumes, to obtain $\mathrm{vol}_{n-b}(S)\,\mathrm{vol}_b(J_1(X), \|\cdot\|_E) \leq \mathrm{vol}_n(X)$, completing the proof. ∎

Proof of Theorem 4.1: We apply Theorem 2.3 together with the inequality $\mathrm{sys}\,\pi_1^2(g) < \sigma_2 \deg(\mathcal{A}_X)$. Here Pu's inequality does not suffice. Indeed, a typical fiber of $\mathcal{A}_X$ may not be diffeomorphic to $\mathbb{R}P^2$. An application of Pu's inequality (3.1) yields a suitably short loop which is essential in the typical fiber. However, a loop which is essential in the typical fiber may not be essential in the ambient manifold X. Thus, we need a generalisation of Pu's inequality. The required generalisation is inequality (3.4) above, cf. [CK03], applied to the composed map $\hat{\phi}: S \to \overline{X} \to K$. ∎

ACKNOWLEDGEMENT: We are grateful to J. Fu for help with integral currents and slices in Section 8.

References

[BaK98] I. Babenko and M. Katz, *Systolic freedom of orientable manifolds,* Annales Scientifiques de l'École Normale Supérieure (Paris) **31** (1998), 787–809.

[BCIK] V. Bangert, C. Croke, S. Ivanov and M. Katz, *Boundary case of equality in optimal Loewner-type inequalities and the filling area conjecture,* in preparation.

[BK03$_A$] V. Bangert and M. Katz, *Stable systolic inequalities and cohomology products,* Communications on Pure and Applied Mathematics **56** (2003), 979–997.

[BK03$_B$] V. Bangert and M. Katz, *An optimal Loewner-type systolic inequality and harmonic one-forms of constant norm,* arXiv:math.DG/0304494.

[Bar57] E. S. Barnes, *On a theorem of Voronoi,* Proceedings of the Cambridge Philosophical Society **53** (1957), 537–539.

[BI94] D. Burago and S. Ivanov, *Riemannian tori without conjugate points are flat,* Geometric and Functional Analysis **4** (1994), 259–269.

[BI95] D. Burago and S. Ivanov, *On asymptotic volume of tori,* Geometric and Functional Analysis **5** (1995), 800–808.

[Ch93] I. Chavel, *Riemannian Geometry—A Modern Introduction,* Cambridge Tracts in Mathematics, **108**,Cambridge University Press, Cambridge, 1993.

[CK03] C. Croke and M. Katz, *Universal volume bounds in Riemannian manifolds,* Surveys in Differential Geometry 8, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeck at Harvard University, May 3–5, 2002 (S. T. Yau, ed.), International Press, Somerville, MA, 2003, pp. 109–137.

[FK92] H. M. Farkas and I. Kra, *Riemann Surfaces,* Second edition, Graduate Texts in Mathematics **71**, Springer-Verlag, New York, 1992.

[Fe69] H. Federer, *Geometric Measure Theory,* Springer, Berlin, 1969.

[Gr81] M. Gromov, *Structures métriques pour les variétés riemanniennes* (J. Lafontaine and P. Pansu, eds.), Textes Mathématiques, 1, CEDIC, Paris, 1981.

[Gr83] M. Gromov, *Filling Riemannian manifolds*, Journal of Differential Geometry **18** (1983), 1–147.

[Gr96] M. Gromov, *Systoles and intersystolic inequalities*, in *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, Séminaires et Congrès, Vol. 1, Société Mathématique de France, Paris, 1996, pp. 291–362.

[Gr99] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, Vol. 152, Birkhäuser, Boston, 1999.

[Iv02] S. Ivanov, *On two-dimensional minimal fillings*, St. Petersburg Mathematical Journal **13** (2002), 17–25.

[Ka03] M. Katz, *Four-manifold systoles and surjectivity of period map*, Commentarii Mathematici Helvetici **78** (2003), 772–876.

[KKS] M. Katz, M. Kreck and A. Suciu, *Free abelian covers, short loops, stable length, and systolic inequalities*,

[LLS90] J. C. Lagarias, H. W. Lenstra Jr. and C. P. Schnorr, *Bounds for Korkin-Zolotarev reduced bases and successive minima of a lattice and its reciprocal lattice*, Combinatorica **10** (1990), 343–358.

[Li69] A. Lichnerowicz, *Applications harmoniques dans un tore*, Comptes Rendus de l'Académie des Sciences, Paris, Série I **269** (1969), 912–916.

[MS86] V. D. Milman and G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces* (with an appendix by M. Gromov), Lecture Notes in Mathematics **1200**, Springer-Verlag, Berlin, 1986.

[NR02] A. Nabutovsky and R. Rotman, *The length of the shortest closed geodesic on a 2-dimansional sphere*, International Mathematics Research Notices 2002:23 (2002), 1211–1222.

[Pu52] P. M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific Journal of Mathematics **2** (1952), 55–71.

[Sa04] S. Sabourau, *Filling radius and short closed geodesics of the two-sphere*, Bulletin de la Société Mathématique de France, to appear.